

# Torus quotients of homogeneous spaces- minimal dimensional Schubert Varieties admitting semi-stable points.

S.S.Kannan, S.K.Pattanayak

Chennai Mathematical Institute, Plot H1, SIPCOT IT Park,  
Padur Post Office, Siruseri, Tamilnadu - 603103, India.  
kannan@cmi.ac.in, santosh@cmi.ac.in

July 31, 2008

30.07.2008

## Abstract

In this paper, for any simple, simply connected algebraic group  $G$  of type  $B_n, C_n$  or  $D_n$  and for any maximal parabolic subgroup  $P$  of  $G$ , we describe all minimal dimensional Schubert varieties in  $G/P$  admitting semistable points for the action of a maximal torus  $T$  with respect to an ample line bundle on  $G/P$ . In this paper, we also describe, for any semi-simple simply connected algebraic group  $G$  and for any Borel subgroup  $B$  of  $G$ , all Coxeter elements  $\tau$  for which the Schubert variety  $X(\tau)$  admits a semistable point for the action of the torus  $T$  with respect to a non-trivial line bundle on  $G/B$ .

Keywords: Semistable points, line bundle, Coxeter element.

## 1 Introduction

Let  $G$  be a simply connected semi-simple algebraic group over an algebraic closed field  $k$ . Let  $T$  be a maximal torus of  $G$  and let  $B$  be a Borel subgroup of  $G$  containing  $T$ . In [4] and [5], the parabolic subgroups  $Q$  of  $G$  containing  $B$  for which there exists an ample line bundle  $\mathcal{L}$  on  $G/Q$  such that the semistable points  $(G/Q)_T^{ss}(\mathcal{L})$  are the same as the stable points  $(G/Q)_T^s(\mathcal{L})$ .

In [7], when  $Q$  is a maximal parabolic subgroup of  $G$  and  $\mathcal{L} = \mathcal{L}_\varpi$ , where  $\varpi$  is a minuscule dominant weight, it is shown that there exists unique minimal dimensional Schubert variety  $X(w)$  admitting semistable points with respect to  $\mathcal{L}$ .

Now, let  $G$  be a simple algebraic group of type  $B, C$  or  $D$  and let  $P$  be a maximal parabolic subgroup of  $G$ . Let  $\mathcal{L}$  be an ample line bundle on  $G/P$ . In this paper, we describe all minimal dimensional Schubert varieties in  $G/P$  admitting semistable points with respect to  $\mathcal{L}$ . For a precise statement, see theorem 3.2.

Now, let  $G$  be a semi-simple simply connected algebraic group over an algebraic closed field  $k$ . Let  $T$  be a maximal torus of  $G$  and let  $B$  be a Borel subgroup of  $G$  containing  $T$ . A Schubert variety  $X(w)$  in  $G/B$  contains a  $(\text{rank } G)$ -dimensional  $T$ -orbit if and only if  $w \geq \tau$  for some Coxeter element  $\tau$ .

So, it is a natural question to ask if for every Coxeter element  $\tau$ , there is a non-trivial line bundle  $\mathcal{L}$  on  $G/B$  such that  $X(w)_T^{ss}(\mathcal{L}) \neq \emptyset$ .

In this paper, we describe all Coxeter elements  $\tau$  for which there exists a non-trivial line bundle  $\mathcal{L}$  on  $G/B$  such that  $X(w)_T^{ss}(\mathcal{L}) \neq \emptyset$ .

The layout of the paper is as follows:

Section 2 consists of preliminary notation and a combinatorial lemma.

Section 3 consists of minimal dimensional Schubert varieties in  $G/P$ , ( where  $G$  is a semi-simple algebraic group of type  $B_n, C_n$  or  $D_n$  and  $P$  is a maximal parabolic subgroup of  $G$ ), admitting semistable points with respect to an ample line bundle on  $G/P$ .

Section 4 consists of description of Coxeter elements for which the corresponding Schubert varieties admit semistable points with respect to a non-trivial line bundle on  $G/B$ .

## 2 Preliminary notation and a combinatorial lemma

This section consists of preliminary notation and a lemma describing a criterion for a Schubert variety to admit semistable points. Let  $G$  be a semi-simple algebraic group over an algebraically closed field  $k$ . Let  $T$  be a maximal torus of  $G$ ,  $B$  a Borel subgroup of  $G$  containing  $T$  and let  $U$  be the unipotent radical of  $B$ . Let  $N_G(T)$  be the normaliser of  $T$  in  $G$ . Let  $W = N_G(T)/T$  be Weyl group of  $G$  with respect to  $T$  and  $R$  denote the set of roots with respect to  $T$ ,  $R^+$  positive roots with respect to  $B$ . Let  $U_\alpha$  denote the one dimensional  $T$ -stable subgroup of  $G$  corresponding to the root  $\alpha$  and let  $S = \{\alpha_1, \dots, \alpha_l\} \subseteq R^+$  denote the set of simple roots. For a subset  $I \subseteq S$  denote  $W^I = \{w \in W | w(\alpha) > 0, \alpha \in I\}$  and  $W_I$  is the subgroup of  $W$  generated by the simple reflections  $s_\alpha, \alpha \in I$ . Then every  $w \in W$  can be uniquely expressed as  $w = w^I \cdot w_I$ , with  $w^I \in W^I$  and  $w_I \in W_I$ . Denote  $R(w) = \{\alpha \in R^+ : w(\alpha) < 0\}$  and  $w_0$  is the longest element of  $W$  with respect to  $S$ . Let  $X(T)$  (resp.  $Y(T)$ ) denote the set of characters of  $T$  (resp. one parameter subgroups of  $T$ ). Let  $E_1 := X(T) \otimes \mathbb{R}$ ,  $E_2 = Y(T) \otimes \mathbb{R}$ . Let  $\langle \cdot, \cdot \rangle : E_1 \times E_2 \longrightarrow \mathbb{R}$  be the canonical non-degenerate bilinear form.

Choose  $\lambda_j$ 's in  $E_2$  such that  $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$  for all  $i$ . Let  $\bar{C} := \{\lambda \in E_2 | \langle \lambda, \alpha \rangle \geq 0 \ \forall \alpha \in R^+\}$  and for all  $\alpha \in R$ , there is a homomorphism  $SL_2 \xrightarrow{\phi_\alpha} G$ , see page-19 of [1]. We have  $\check{\alpha} : G_m \longrightarrow G$  defined by  $\check{\alpha}(t) = \phi_\alpha\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right)$ . We also have  $s_\alpha(\chi) = \chi - \langle \chi, \check{\alpha} \rangle \alpha$  for all  $\alpha \in R$  and  $\chi \in E_1$ . Set  $s_i = s_{\alpha_i} \ \forall \ i = 1, 2, \dots, l$ . Let  $\{\omega_i : i = 1, 2, \dots, l\} \subset E_1$  be the fundamental weights; i.e.  $\langle \omega_i, \check{\alpha}_j \rangle = \delta_{ij}$  for all  $i, j = 1, 2, \dots, l$ .

For any character  $\chi$  of  $B$ , we denote by  $\mathcal{L}_\chi$ , the line bundle on  $G/B$  given by the character  $\chi$ . Let  $X(w) = \overline{BwB/B}$  denote the Schubert variety corresponding to  $w$ . We denote by  $X(w)_T^{ss}(\mathcal{L}_\chi)$  the semistable points of  $X(w)$  for the action of  $T$  with respect to the line bundle  $\mathcal{L}_\chi$ .

**Lemma 2.1.** *Let  $\chi = \sum_{\alpha \in S} a_\alpha \varpi_\alpha$  be a dominant character of  $T$  which is in the root lattice. Let  $I = \text{Supp}(\chi) = \{\alpha \in S : a_\alpha \neq 0\}$  and let  $w \in W^{I^c}$ . Then  $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$  if and only if  $w\chi \leq 0$ .*

*Proof.* If  $X(w)_T^{ss}(\mathcal{L}_\chi) \neq \emptyset$ , then, by Hilbert-Mumford criterion (Theorem (2.1) of [8]) and lemma (2.1) of [10], we see that  $w\chi \leq 0$ .

Conversely, let  $w\chi \leq 0$ .

Step 1 - We prove that if  $w, \tau \in W^{I^c}$  are such that  $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$ , then,  $w \leq \tau$ . Now, suppose that  $X(w) \subseteq \bigcup_{\phi \in W} \phi X(\tau)$ . Then, since  $X(w)$  is irreducible and  $W$  is finite, we must have

$$X(w) \subseteq \phi X(\tau), \text{ for some } \phi \in W.$$

Hence,  $\phi^{-1}X(w) \subseteq X(\tau)$ . Now, let  $P_I = BW_I B$  and consider the projection

$$\pi : G/B \longrightarrow G/P_I$$

Then,  $\pi^{-1}(\phi^{-1}X(w)) \subseteq \pi^{-1}(X(\tau))$ . Let  $w^{max}$  (resp.  $\tau^{max}$ ) be the maximal representative of  $w$  (resp.  $\tau$ ) in  $W$ .

Hence,  $\phi^{-1}X(w^{max}) \subseteq X(\tau^{max})$ . So, we may assume that  $I = S$ .

Now, since  $\phi^{-1}X(w) \subseteq X(\tau)$ , we have  $\phi^{-1}w_1 \leq \tau, \ \forall \ w_1 \leq w$ .

Therefore  $w_1 \phi \leq \tau^{-1} \ \forall \ w_1 \leq w^{-1}$ . Hence, by lemma (5.6) of [6], we have  $\tau^-(w^{-1}, \phi^{-1})\phi \leq \tau^{-1}$ .

Hence,  $w^{-1} \leq \tau^-(w^{-1}, \phi^{-1})\phi \leq \tau^{-1}$ . So  $w \leq \tau$ .

Now, let  $w \in W^{I^c}$  be such that  $w\chi \leq 0$ . Then by step 1, there exist a point  $x \in X(w) \setminus W$ -translates of  $X(\tau), \tau \in W^{I^c}, \tau \not\leq w$ .  $\longrightarrow (1)$ .

Step 2: We prove that  $x$  is semistable.

Let  $\lambda$  be an one parameter subgroup of  $T$ . Choose  $\phi \in W$  such that  $\phi\lambda \in \bar{C}$ . Let  $\tau \in W^{I^c}$  be such that  $\phi x \in U_\tau \tau P_I$ .

By (1),  $w \leq \tau$ . Hence,  $\tau\chi \leq w\chi \leq 0$ .

Hence, by lemma (2.1) of [10], we have  $\mu^L(x, \lambda) = \mu^L(\phi x, \phi\lambda) = \langle -\tau\chi, \lambda \rangle \geq 0$ .

Hence, by Hilbert-Mumford criterion (Theorem (2.1) of [8]),  $x$  is semistable.

□

### 3 Minimal dimensional Schubert variety in $G/P$ admitting semistable points

In this section, we describe all minimal dimensional Schubert varieties  $X(w)$  in  $G/P$  (where  $G$  is a simple algebraic group of type  $B$ ,  $C$  or  $D$ , and  $P$  is a maximal parabolic subgroup of  $G$ ) for which  $X(w)$  admits a semistable point for the action of a maximal torus of  $G$  with respect to an ample line bundle on  $G/P$ .

Let  $I_r = S \setminus \{\alpha_r\}$  and let  $P_{I_r} = BW_{I_r}B$  be the maximal parabolic corresponding to the simple root  $\alpha_r$ . In this section we will describe all minimal elements of  $W^{I_r}$  for which  $X(w)_T^{ss}(\mathcal{L}_r) \neq \emptyset$ .

At this point, we recall a standard property of the fundamental weights of type  $A_n, B_n, C_n$  and  $D_n$ .

In types  $A_n, B_n, C_n$  and  $D_n$ , we have  $|\langle \varpi_r, \check{\alpha} \rangle| \leq 2$  for any fundamental weight  $\varpi_r$  and any root  $\alpha$ .

*Proof.* Now  $\langle \varpi_r, \check{\alpha} \rangle \leq \langle \varpi_r, \check{\eta} \rangle$ , where  $\eta$  is a highest root for the corresponding root system.

The highest root for type  $A_n$  is  $\alpha_1 + \alpha_2 + \dots + \alpha_n$ , the highest roots for type  $B_n$  are  $\alpha_1 + 2(\alpha_2 + \dots + \alpha_n)$  and  $\alpha_1 + \alpha_2 + \dots + \alpha_n$ , the highest roots for type  $C_n$  are  $2(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}) + \alpha_n$  and  $\alpha_1 + 2(\alpha_2 + \dots + \alpha_{n-1}) + \alpha_n$  and the unique highest root for type  $D_n$  is  $\alpha_1 + 2(\alpha_2 + \dots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$ .

In all these cases, we have  $\langle \varpi_r, \check{\eta} \rangle \leq 2$ . So  $|\langle \varpi_r, \check{\alpha} \rangle| \leq 2$ , for any root  $\alpha$ . □

Let  $G$  be a simple simply-connected algebraic group of type  $B_n, C_n$  or  $D_n$ . Let  $T$  be a maximal torus of  $G$  and let  $S$  be the set of simple roots with respect to a Borel subgroup  $B$  of  $G$  containing  $T$ .

**Proposition 3.1.** *Let  $I_r = S \setminus \{\alpha_r\}$  and let  $w \in W^{I_r}$  be of maximal length such that  $w(\varpi_r) \geq 0$ . Write  $w(\varpi_r) = \sum_{i=1}^n a_i \alpha_i$  and let  $a = \max\{a_i : i = 1, 2, \dots, n\}$ . Then  $a \in \{1, \frac{3}{2}\}$ . Further, if  $a = \frac{3}{2}$ , then  $r$  must be odd and  $\varpi_r$  must be in type  $D_n$  and  $a = a_{n-1}$  or  $a = a_n$ .*

*Proof.* Since  $2 \leq r \leq n - 2$ , we have  $2\varpi_r \in Z_{\geq 0}S$ . Hence, if  $a \in \{1, \frac{3}{2}\}$ , then  $a \geq 2$ .

Let  $i_0$  be the least integer such that  $a_{i_0} = a$ .

Clearly,  $i_0 \neq 1$ . We first observe that,  $s_{i_0}w(\varpi_r) = w(\varpi_r) - \langle w(\varpi_r), \check{\alpha}_{i_0} \rangle \alpha_{i_0} \geq 0$ , since,  $\langle w(\varpi_r), \check{\alpha}_{i_0} \rangle \leq 2 \leq a = a_{i_0}$ .

For all the cases except  $i_0 = n$  in type  $B_n$ ,  $i_0 = n - 1$  in type  $C_n$  and  $i_0 = n - 2, n - 1, n$  in type  $D_n$ , we have  $\langle w(\varpi_r), \check{\alpha}_{i_0} \rangle = 2a - (a_{i_0-1} + a_{i_0+1}) > 0$ . Hence,  $s_{\alpha_{i_0}}w(\varpi_r) < w(\varpi_r)$ . So,  $s_{\alpha_{i_0}}w > w$ , a contradiction to the maximality of  $w$ .

Now, we treat the special cases explicitly.

*Case 1 :  $i_0 = n$  in type  $B_n$ .*

In this case,  $\langle w(\varpi_r), \check{\alpha}_n \rangle = -2a_{n-1} + 2a_n > 0$ , since  $a_n = a > a_{n-1}$ . So,  $s_nw(\varpi_r) < w(\varpi_r)$ . Hence,  $s_nw > w$ , a contradiction to the maximality of  $w$ .

*Case 2 :  $i_0 = n - 1$  in type  $C_n$ .*

In this case  $\langle w(\varpi_r), \check{\alpha}_{n-1} \rangle = -a_{n-2} + 2a_{n-1} - 2a_n$ .

So, we need to show that  $2a_{n-1} > a_{n-2} + 2a_n$ . If not, then  $2a_n \geq a_{n-1} + 1$ , since  $a_{n-2} \leq a_{n-1} - 1$ .

Now, we have  $s_nw(\varpi_r) = \sum_{i \neq n} a_i \alpha_i + (a_{n-1} - a_n) \alpha_n \geq 0$ , since  $a_{n-1} = a \geq a_n$ .

On the other hand, since  $2a_n \geq a_{n-1} + 1$ , we have  $a_{n-1} - a_n \leq a_n - 1$ . So,  $s_nw(\varpi_r) < w(\varpi_r)$ . Hence,  $s_nw > w$ , a contradiction to the maximality of  $w$ .

*Case 3 :  $i_0 = n$  in type  $D$ .*

Here, we have  $\langle w(\varpi_r), \check{\alpha}_n \rangle = 2a_n - a_{n-2} > 0$ , since  $a_n = a > a_{n-2}$ .

So,  $s_nw(\varpi_r) < w(\varpi_r)$ . Hence,  $s_nw > w$ , a contradiction to the maximality of  $w$ .

*Case 4 :  $i_0 = n - 1$  in type  $D$ .*

This case is similar to Case-3.

*Case 5 :  $i_0 = n - 2$  in type  $D$ .*

We have  $\langle w(\varpi_r), \check{\alpha}_{n-2} \rangle = -a_{n-3} + 2a_{n-2} - a_{n-1} - a_n$ .

In order to prove that  $\langle w(\varpi_r), \check{\alpha}_{n-2} \rangle > 0$ , we need to prove  $a_{n-1} + a_n \leq a_{n-2}$ , since  $a_{n-3} < a_{n-2}$ .

Suppose  $a_{n-1} + a_n \geq a_{n-2} + 1$ . Then, we have either  $2a_{n-1} > a_{n-2}$  or  $2a_n > a_{n-2}$ .

Without loss of generality, we may assume that  $2a_{n-1} > a_{n-2}$ . Hence we have

$s_{n-1}w(\varpi_r) = \sum_{i \neq n-1} a_i \alpha_i + (a_{n-2} - a_{n-1}) \alpha_{n-1} \leq w(\varpi_r)$ , since  $a_{n-2} - a_{n-1} < a_{n-1}$ .

On the other hand,  $s_{n-1}w(\varpi_r) \geq 0$ , since  $a_{n-2} = a \geq a_{n-1}$ . So,  $s_{n-1}w > w$ , a contradiction to the maximality of  $w$ .

Thus, we conclude that  $a \in \{1, \frac{3}{2}\}$ .

Now, if  $a = \frac{3}{2}$ , then clearly  $r$  is odd and  $G$  is not of type  $B_n$ . We now prove that  $G$  can not be of type  $C_n$ .

Suppose on the contrary, let  $t$  be the least positive integer such that  $\sum_{i=t}^{n-1} \alpha_i + \frac{3}{2}\alpha_n \leq w(\varpi_r)$ .

Since  $\langle w(\varpi_r), \check{\alpha}_n \rangle = 3 - a_{n-1} \leq 2$ , we have  $a_{n-1} = 1$ .

If  $t \leq n-2$ , then  $0 \leq s_t w(\varpi_r) = \sum_{i \neq t} a_i \alpha_i < w(\varpi)$ . So,  $s_t w > w$ , a contradiction to the maximality of  $w$ . Hence,  $a_{n-2} = 0$ .

We now claim that  $a_i = 0 \forall i \leq n-3$ . For otherwise, let  $m \leq n-3$  be the largest integer such that  $a_m = 1$ .

Now,  $\langle w(\varpi_r), \alpha_{m+1} + \alpha_{m+2} + \dots + \alpha_{n-1} \rangle = -3$ , a contradiction to the fact that  $|\langle w(\varpi_r), \check{\beta} \rangle| \leq 2$  for all root  $\beta$ .

Thus,  $a_i = 0 \forall i \leq n-2$ . Hence,  $w(\varpi) = \alpha_{n-1} + \frac{3}{2}\alpha_n$ . But  $\langle w(\varpi_r), \alpha_{n-1} + \alpha_n \rangle = 3$ , a contradiction to the fact that  $|\langle w(\varpi_r), \check{\beta} \rangle| \leq 2$  for all root  $\beta$ .

Thus,  $G$  can not be of type  $C_n$ .

If  $G$  is of type  $D_n$ , then  $a_i \leq \frac{3}{2} \forall i = 1, 2, \dots, n$ . We now claim that,  $a_{n-1} + a_n \leq 2$ . Suppose on the contrary, let  $a_{n-1} = a_n = \frac{3}{2}$ .

We claim that  $a_m = 0 \forall m \leq n-3$ . Otherwise, let  $t$  be the least positive integer such that  $\sum_{i=t}^{n-2} \alpha_i + \frac{3}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n \leq w(\varpi_r)$ . Then,  $a_{t-1} = 0$  and  $t \leq n-3$ .

Hence,  $\langle w(\varpi_r), \alpha_t + \alpha_{t+1} + \dots + \alpha_{n-1} + \alpha_n \rangle = 3$ , a contradiction to the fact that  $|\langle w(\varpi_r), \check{\beta} \rangle| \leq 2$  for all root  $\beta$ .

Thus,  $a_m = 0 \forall m \leq n-3$ . Hence,  $w(\varpi) = \alpha_{n-2} + \frac{3}{2}(\alpha_{n-1} + \alpha_n)$ .

So,  $\langle w(\varpi_r), \alpha_{n-2} + \alpha_{n-1} + \alpha_n \rangle = 3$ , a contradiction to the fact that  $|\langle w(\varpi_r), \check{\beta} \rangle| \leq 2$  for all root  $\beta$ .

Thus, in type  $D_n$  not both  $a_{n-1}$  and  $a_n$  can be  $\frac{3}{2}$ .

□

Notation:  $J_{p,q} = \{(i_1, i_2, \dots, i_p) : i_k \in \{1, 2, \dots, q\} \forall k \text{ and } i_{k+1} - i_k \geq 2\}$

Now, we will describe the set of all elements  $w \in W^{I_r}$  of minimal length such that  $w\varpi_r \leq 0$  for types  $B_n, C_n$  and  $D_n$ .

**Theorem 3.2.** Let  $W_{min}^{I_r} =$  Minimal elements of the set of all  $\tau \in W^{I_r}$  such that  $X(\tau)_T^{ss}(\mathcal{L}_{\varpi_r}) \neq \emptyset$ .

(1) Type  $B_n$ : (i) Let  $r = 1$ . Then  $w = s_n s_{n-1} \dots s_1$ .

(ii) Let  $r$  be an even integer in  $\{2, 3, \dots, n\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2}, n-1}\}$ .

(iii) Let  $r$  be an odd integer in  $\{2, 3, \dots, n\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \alpha_n)$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} :$

$\underline{i} \in J_{\frac{r-1}{2}, n-2}$ .

(iv) Let  $r = n$ . If  $n$  is even, then,  $w = w_{\frac{n}{2}} \cdots w_1$ , where,  $w_i = s_{2i-1} \cdots s_n$ ,  $i = 1, 2, \dots, \frac{n}{2}$  and if  $n$  is odd, then,  $w = w_{[\frac{n}{2}]+1} \cdots w_1$ , where,  $w_i = s_{2i-1} \cdots s_n$ ,  $i = 1, 2, \dots, [\frac{n}{2}] + 1$ .

(2) Type  $C_n$ : (i) Let  $r = 1$ . Then  $w = s_n s_{n-1} \cdots s_1$ .

(ii) Let  $r$  be an even integer in  $\{2, 3, \dots, n-1\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2}, n-1}\}$ .

(iii) Let  $r$  be an odd integer in  $\{2, 3, \dots, n-1\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_n)$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r-1}{2}, n-2}\}$ .

(3) Type  $D_n$ : (i) Let  $r = 1$ . Then  $w = s_n s_{n-1} \cdots s_1$ .

(ii) Let  $r$  be an even integer in  $\{2, 3, \dots, n-2\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n} \setminus Z$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ , where  $Z = \{(i_1, i_2, \dots, i_{\frac{r}{2}-2}, n-2, n) : i_k \in \{1, 2, \dots, n-4\} \text{ and } i_{k+1} - i_k \geq 2 \forall k\}$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2}, n} \setminus Z\}$ .

(iii) Let  $r$  be an odd integer in  $\{2, 3, \dots, n-2\}$ . For any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-3}$ , there exists unique  $w_{\underline{i}} \in W_{min}^{I_r}$  such that  $w_{\underline{i}}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$ . Also, for any  $\underline{i} = (i_1, i_2, \dots, i_{\frac{r-1}{2}}) \in J_{\frac{r-1}{2}, n-2}$ , there exists unique  $w_{\underline{i},1} \in W_{min}^{I_r}$  such that  $w_{\underline{i},1}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{1}{2}\alpha_{n-1} + \frac{3}{2}\alpha_n)$  and there exists unique  $w_{\underline{i},2} \in W_{min}^{I_r}$  such that  $w_{\underline{i},2}(\varpi_r) = -(\sum_{k=1}^{\frac{r-1}{2}} \alpha_{i_k} + \frac{3}{2}\alpha_{n-1} + \frac{1}{2}\alpha_n)$ . Further,  $W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r-1}{2}, n-3}\} \cup \{w_{\underline{i},j} : \underline{i} \in J_{\frac{r-1}{2}, n-2} \text{ and } j = 1, 2\}$ .

(iv) Let  $r = n-1$  or  $n$ . Then,  $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$ , where,

$$w_i = \begin{cases} \tau_i s_n & \text{if } i \text{ is odd.} \\ \tau_i s_{n-1} & \text{if } i \text{ is even.} \end{cases}$$

with,  $\tau_i = s_{2i-1} \cdots s_{n-2}$ ,  $i = 1, 2, \dots, [\frac{n-1}{2}]$ .

Proof. Proof of 1:

(i)  $\varpi_1 = \alpha_1 + \alpha_2 + \dots + \alpha_n$ .

Take  $w = s_n s_{n-1} \cdots s_1$ . Then  $w(\varpi_1) = -\alpha_n \leq 0$ .

(ii) Let  $r$  be an even integer in  $\{2, 3, \dots, n-2\}$ .

We have,  $\varpi_r = \sum_{i=1}^{r-1} i\alpha_i + r(\alpha_r + \dots + \alpha_n)$ ,  $4 \leq r \leq (n-1)$ .

Now,  $J_{\frac{r}{2}, n-1} = \{(i_1, i_2, \dots, i_{\frac{r}{2}}) : i_k \in \{1, 2, \dots, n-1\} \text{ and } i_{k+1} - i_k \geq 2 \forall k\}$ . Consider the partial order on  $J_{\frac{r}{2}, n-1}$ , given by  $(i_1, i_2, \dots, i_{\frac{r}{2}}) \leq (j_1, j_2, \dots, j_{\frac{r}{2}})$  if  $i_k \leq j_k \forall k$  and

$(i_1, i_2, \dots, i_{\frac{r}{2}}) < (j_1, j_2, \dots, j_{\frac{r}{2}})$  if  $i_k < j_k$  for some  $k$ . We will prove the theorem by induction on this order.

For  $(j_1, j_2, \dots, j_{\frac{r}{2}}) = (n-r+1, n-r+3, \dots, n-1)$ , we have

$$(s_{n-r+1} \dots s_1)(s_{n-r+3} \dots s_2) \dots (s_{n-1} \dots s_{\frac{r}{2}})(s_n s_{n-1} \dots s_{\frac{r}{2}+1})(s_n s_{n-1} \dots s_{\frac{r}{2}+2}) \dots (s_n s_{n-1} \dots s_r)(\varpi_r) = -(\sum_{t=1}^{\frac{r}{2}} \alpha_{n-r+2t-1}).$$

Now, if  $(i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$  is not maximal, then, there exists  $t$  maximal such that  $i_t < n-r+2t-1$ .

Now,  $(i_1, i_2, \dots, i_{t-1}, 1+i_t, i_{t+1}, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$  and  $(i_1, i_2, \dots, i_{t-1}, 1+i_t, i_{t+1}, \dots, i_{\frac{r}{2}}) > (i_1, i_2, \dots, i_{\frac{r}{2}})$ . So, by induction, there exists  $w_1 \in W^{I_r}$  such that  $w_1 \varpi_r = -(\sum_{k \neq t} \alpha_{i_k} + \alpha_{1+i_t})$ . Taking  $w = s_{1+i_t} s_{i_t} w_1$  we have  $w \varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ .

Hence, for any  $(i_1, i_2, \dots, i_{\frac{r}{2}}) \in J_{\frac{r}{2}, n-1}$ , there exists  $w \in W^{I_r}$  of minimal length such that  $w \varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ .

Now, we will prove that the  $w$ 's in  $W^{I_r}$  having this property are minimal.

Let  $w \in W^{I_r}$  such that  $w \varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ .

Suppose  $w$  is not minimal. Then there exist  $\beta \in R^+$  such that  $s_\beta w(\varpi_r) \leq 0$  and  $l(s_\beta w) = l(w) - 1$ . Since  $s_\beta w(\varpi_r) \leq 0$ , and  $i_{k+1} - i_k \geq 2 \forall k$ ,  $\beta = \alpha_{i_k}$  for some  $k = 1, 2, \dots, \frac{r}{2}$ .

Since  $l(s_\beta w) = l(w) - 1$ ,  $\beta = \alpha_{i_t}$  for some  $t$ . Hence,  $s_\beta w(\varpi_r) = -(\sum_{k \neq t} \alpha_{i_k} + \alpha_{i_t}) \not\leq 0$ , a contradiction. Thus, all the  $w$ 's are minimal.

Now, it remains to prove that for all elements of the type  $-(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$  in the weight lattice such that  $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ , for some  $k$ , there does not exist  $w \in W^{I_r}$ , of minimal length such that  $w \varpi_r = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ .

Let  $\mu = -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$  be such that  $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$  for some  $k$ . Choose  $k$  minimal such that  $\langle \alpha_{i_k}, \alpha_{i_{k+1}} \rangle \neq 0$ .

If  $i_k = n-1$ , then  $i_{k+1} = 1$  and  $s_n w(\varpi_n) = -(\sum_{i_j \neq n} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ . Hence,  $s_n w < w$ , a contradiction to the minimality of  $w$ .

Otherwise,  $s_{i_k} w(\varpi_r) = -(\sum_{j \neq k} \alpha_{i_j}) > -(\sum_{k=1}^{\frac{r}{2}} \alpha_{i_k})$ . Hence,  $s_{i_k} w < w$ , a contradiction to the minimality of  $w$ .

(iii) Let  $r$  be an odd integer in  $\{2, 3, \dots, n-1\}$ .

The proof is similar the case when  $r$  is even.

(iv) We have,  $\varpi_n = \frac{1}{2} \sum_{i=1}^n i \alpha_i$ .

Then,  $2\varpi_n = \sum_{i=1}^n i \alpha_i$ .

Case 1 :  $n$  is even.

Take  $w_i = s_{2i-1} \dots s_n$ ,  $i = 1, 2, \dots, \frac{n}{2}$ .



Let  $w = w_{\frac{n}{2}} \cdots w_1$ . Then  $w(2\varpi_n) = -\sum_{i=1}^{\frac{n}{2}} \alpha_{2i-1} \leq 0$ .

$W_{min}^{I_r} = \{w_{\underline{i}} : \underline{i} \in J_{\frac{r}{2}, n-1}\}$  follows from lemma (2.1).

Case 2 :  $n$  is odd.

Take  $w_i = s_{2i-1} \cdots s_n$ ,  $i = 1, 2, \dots, \frac{n+1}{2}$ .

Let  $w = w_{\frac{n+1}{2}} \cdots w_1$ . Then  $w(2\varpi_n) = -\sum_{i=1}^{\frac{n+1}{2}} \alpha_{2i-1} \leq 0$ .

Proof of 2:

(i) We have,  $\varpi_1 = \alpha_1 + \alpha_2 + \dots + \frac{1}{2}\alpha_n$ .

Then,  $2\varpi_1 = 2(\alpha_1 + \alpha_2 + \dots + \alpha_{n-1}) + \alpha_n$ .

Take  $w = s_n s_{n-1} \cdots s_1$ . Then  $w(2\varpi_1) = -\alpha_n \leq 0$ .

Proof of (ii) and (iii) are similar to Cases (ii) and (iii) of type  $B$ .

Proof of 3:

(i) We have,  $\varpi_1 = \sum_{i=1}^{n-2} \alpha_i + \frac{1}{2}(\alpha_{n-1} + \alpha_n)$ .

Then,  $2\varpi_1 = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$ .

Take  $w = s_n s_{n-1} \cdots s_1$ . Then  $w(2\varpi_1) = -(\alpha_{n-1} + \alpha_n) \leq 0$ .

Proof of (ii) and (iii) are very similar to Cases (ii) and (iii) of type  $B$ .

(iv) We have,  $\varpi_{n-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + \frac{1}{4}(n\alpha_{n-1} + (n-2)\alpha_n)$ .

Then,  $4\varpi_{n-1} = 2(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + n\alpha_{n-1} + (n-2)\alpha_n$

Take

$$w_i = \begin{cases} \tau_i s_{n-1} & \text{if } i \text{ is odd.} \\ \tau_i s_n & \text{if } i \text{ is even.} \end{cases}$$

where,  $\tau_i = s_{2i-1} \cdots s_{n-2}$ ,  $i = 1, 2, \dots, [\frac{n-1}{2}]$ .

Let  $w = \prod_{i=1}^{[\frac{n-1}{2}]} w_i$ . Then,

$$w(4\varpi_{n-1}) = \begin{cases} \mu - 2\alpha_n & \text{if } n \equiv 0 \pmod{4}, \\ \mu - 2\alpha_{n-1} & \text{if } n \equiv 2 \pmod{4}, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n & \text{if } n \equiv 1 \pmod{4}, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where,  $\mu = -2(\sum_{i=1}^{[\frac{n-1}{2}]} \alpha_{2i-1})$ .

We have,  $\varpi_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + \frac{1}{4}((n-2)\alpha_{n-1} + n\alpha_n)$ .

Then,  $4\varpi_n = 2(\alpha_1 + 2\alpha_2 + \dots + (n-2)\alpha_{n-2}) + (n-2)\alpha_{n-1} + n\alpha_n$ .

Take

$$w_i = \begin{cases} \tau_i s_n & \text{if } i \text{ is odd.} \\ \tau_i s_{n-1} & \text{if } i \text{ is even.} \end{cases}$$

where,  $\tau_i = s_{2i-1} \dots s_{n-2}$ ,  $i = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$ .

Let  $w = \prod_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} w_i$ . Then,

$$w(4\varpi_n) = \begin{cases} \mu - 2\alpha_{n-1} & \text{if } n \equiv 0 \pmod{4}, \\ \mu - 2\alpha_n & \text{if } n \equiv 2 \pmod{4}, \\ \mu - 2\alpha_{n-2} - \alpha_{n-1} - 3\alpha_n & \text{if } n \equiv 1 \pmod{4}, \\ \mu - 2\alpha_{n-2} - 3\alpha_{n-1} - \alpha_n & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where,  $\mu = -2(\sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} \alpha_{2i-1})$ .

□

## 4 Coxeter elements admitting semistable points

In this section, we describe all Coxeter elements  $w \in W$  for which the corresponding Schubert variety  $X(w)$  admit a semistable point for the action of a maximal torus with respect to a non-trivial line bundle on  $G/B$ .

Now, let us assume that the root system is irreducible, see page 52 of [2].

### Coxeter elements of Weyl group:

An element  $w \in W$  is said to be a Coxeter element if it is of the form  $w = s_{i_1} s_{i_2} \dots s_{i_n}$ , with  $s_{i_j} \neq s_{i_k}$  unless  $j = k$ , see page 74 of [3].

Let  $\chi = \sum_{\alpha \in S} a_\alpha \alpha$  be a non-zero dominant weight and let  $w$  be a Coxeter element of  $W$ .

**Lemma 4.1.** *If  $w\chi \leq 0$  and  $\alpha \in S$  is such that  $l(ws_\alpha) = l(w) - 1$ , then,*

$$(1) |\{\beta \in S \setminus \{\alpha\} : \langle \beta, \check{\alpha} \rangle \neq 0\}| = 1 \text{ or } 2.$$

(2) *Further if  $|\{\beta \in S \setminus \{\alpha\} : \langle \beta, \check{\alpha} \rangle \neq 0\}| = 2$ , then  $R$  must be of type  $A_3$  and  $\chi$  is of the form  $a(2\alpha + \beta + \gamma)$  for some  $a \in \mathbb{Z}_{\geq 0}$ , where  $\alpha, \beta$  and  $\gamma$  are labelled as  $\circ_\beta \text{---} \circ_\alpha \text{---} \circ_\gamma$ .*

*Proof.* Since  $S$  is irreducible and  $\chi$  is non zero dominant weight,  $a_\beta$  is a positive rational number for each  $\beta \in S$ . Further since  $w\chi \leq 0$ ,  $\chi$  must be in the root lattice and so  $a_\beta$  is a positive integer for every  $\beta$  in  $S$ .

Since  $w$  is a Coxeter element and  $l(ws_\alpha) = l(w) - 1$ , the coefficient of  $\alpha$  in  $w\chi =$  coefficient of  $\alpha$  in  $s_\alpha \chi$ . → (1)

$$\begin{aligned}
\text{We have } s_\alpha \chi &= \chi - \langle \chi, \check{\alpha} \rangle \alpha \\
&= \chi - \langle \sum_{\beta \in S} a_\beta \beta, \check{\alpha} \rangle \alpha \\
&= \sum_{\beta \in S} a_\beta \beta - \sum_{\beta \in S} a_\beta \langle \beta, \check{\alpha} \rangle \alpha.
\end{aligned}$$

The coefficient of  $\alpha$  in  $s_\alpha \chi$  is  $-(\sum_{\beta \in S \setminus \{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta + a_\alpha)$ .  $\longrightarrow (2)$

Since  $w\chi \leq 0$ , from (1) and (2) we have

$$-(\sum_{\beta \in S \setminus \{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta + a_\alpha) \leq 0.$$

Hence,  $-(\sum_{\beta \in S \setminus \{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta) \leq a_\alpha$

Thus, we have  $-2(\sum_{\beta \in S \setminus \{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta) \leq 2a_\alpha$ .  $\longrightarrow (3)$

Since  $\chi$  is dominant, we have,

$$\begin{aligned}
\langle \chi, \check{\beta} \rangle &\geq 0, \quad \forall \beta \in S \\
&\Rightarrow \langle \sum_{\gamma \in S} a_\gamma \gamma, \check{\beta} \rangle \geq 0 \\
&\Rightarrow \sum_{\gamma \in S} a_\gamma \langle \gamma, \check{\beta} \rangle \geq 0
\end{aligned}$$

Now if  $\langle \beta, \check{\alpha} \rangle \neq 0$ , the left hand side of the inequality is  $2a_\beta - a_\alpha$  -(a non-negative integer).

Thus, we have,  $2a_\beta \geq a_\alpha$  if  $\langle \beta, \check{\alpha} \rangle \neq 0$   $\longrightarrow (4)$ .

Now if  $|\{\beta \in S \setminus \{\alpha\} : \langle \beta, \check{\alpha} \rangle \neq 0\}| \geq 3$ , from (3) and (4) we have,

$$3a_\alpha \leq -(2 \sum_{\beta \in S \setminus \{\alpha\}} \langle \beta, \check{\alpha} \rangle a_\beta) \leq 2a_\alpha.$$

This is a contradiction to the fact that  $a_\alpha$  is a positive integer.

So  $|\{\beta \in S \setminus \{\alpha\} : \langle \beta, \check{\alpha} \rangle \neq 0\}| \leq 2$ .

### **Proof of (2):**

Suppose  $|\{\beta \in S \setminus \{\alpha\} : \langle \beta, \check{\alpha} \rangle \neq 0\}| = 2$ . Let  $\beta, \gamma$  be the two distinct elements of this set.

Using (3) and the facts  $\langle \beta, \check{\alpha} \rangle \leq -1$ ,  $\langle \gamma, \check{\alpha} \rangle \leq -1$ , we have

$$2(a_\beta + a_\gamma) \leq -2(\langle \beta, \check{\alpha} \rangle a_\beta + \langle \gamma, \check{\alpha} \rangle a_\gamma) \leq 2a_\alpha \quad \longrightarrow (5)$$

Since  $\langle \chi, \check{\beta} \rangle \geq 0$  and  $\langle \chi, \check{\gamma} \rangle \geq 0$  we have

$$2a_\beta \geq -\sum_{\delta \neq \beta, \alpha} \langle \delta, \check{\beta} \rangle a_\delta + a_\alpha \text{ and } 2a_\gamma \geq -\sum_{\delta \neq \gamma, \alpha} \langle \delta, \check{\gamma} \rangle a_\delta + a_\alpha.$$

Hence,  $-\sum_{\delta \neq \beta, \alpha} \langle \delta, \check{\beta} \rangle a_\delta - \sum_{\delta \neq \gamma, \alpha} \langle \delta, \check{\gamma} \rangle a_\delta + 2a_\alpha \leq 2(a_\beta + a_\gamma)$ .

Using (5), we get

$$\begin{aligned}
&-\sum_{\delta \neq \beta, \alpha} \langle \delta, \check{\beta} \rangle a_\delta - \sum_{\delta \neq \gamma, \alpha} \langle \delta, \check{\gamma} \rangle a_\delta + 2a_\alpha \leq 2a_\alpha. \\
&\Rightarrow \sum_{\delta \neq \gamma, \beta, \alpha} \langle -\delta, \check{\beta} \rangle a_\delta + \sum_{\delta \neq \gamma, \beta, \alpha} \langle -\delta, \check{\gamma} \rangle a_\delta \leq 0, \text{ since } \langle \beta, \check{\gamma} \rangle = \langle \gamma, \check{\beta} \rangle = 0
\end{aligned}$$

Since each  $a_\delta$  is positive and  $\langle -\delta, \check{\beta} \rangle, \langle -\delta, \check{\gamma} \rangle$  are non-negative integers, we have

$$\langle -\delta, \check{\beta} \rangle = 0 \text{ and } \langle -\delta, \check{\gamma} \rangle = 0, \forall \delta \neq \alpha, \beta, \gamma.$$

Since  $R$  is irreducible, we have  $S = \{\alpha, \beta, \gamma\}$ . So, from the classification theorem ( see page 57 and 58 of [2]) of irreducible root systems, we have  $\langle \beta, \check{\alpha} \rangle \in \{-1, -2\}$ .

If  $\langle \beta, \check{\alpha} \rangle = -2$ , then  $\langle \gamma, \check{\alpha} \rangle = -1$ .

$$\text{Hence, from (3) we get } 4a_\beta + 2a_\gamma \leq 2a_\alpha \longrightarrow (6)$$

Again, from (4) we have  $2a_\beta \geq a_\alpha$  and  $2a_\gamma \geq a_\alpha$ . So using (6), we get  $3a_\alpha \leq 4a_\beta + 2a_\alpha \leq 2a_\alpha$ , a contradiction to the fact that  $a_\alpha$  is a positive integer. Thus  $\langle \beta, \check{\alpha} \rangle = -1$ .

Using a similar argument, we see that  $\langle \gamma, \check{\alpha} \rangle = -1$ .

Now, let us assume that  $\langle \alpha, \check{\beta} \rangle = -2$ .

Then,

$$\begin{aligned} 0 \leq \langle \chi, \check{\beta} \rangle &= a_\gamma \langle \gamma, \check{\beta} \rangle - 2a_\alpha + 2a_\beta \\ &= -2a_\alpha + 2a_\beta, \text{ since } \langle \gamma, \check{\beta} \rangle = 0 \\ \Rightarrow 2a_\alpha &\leq 2a_\beta. \end{aligned}$$

From (3), we have  $2a_\beta + 2a_\gamma \leq 2a_\alpha \leq 2a_\beta$ .

Hence,  $2a_\gamma \leq 0$ , a contradiction. So  $\langle \alpha, \check{\beta} \rangle = -1$ . Similarly  $\langle \alpha, \check{\gamma} \rangle = -1$ .

Hence  $R$  is of the type  $A_3$ .

$$\circ_\beta \text{---} \circ_\alpha \text{---} \circ_\gamma$$

We now show that  $\chi = a(\beta + 2\alpha + \gamma)$ , for some  $a \in \mathbb{Z}_{\geq 0}$ .

Let  $\chi = a_\alpha \alpha + a_\beta \beta + a_\gamma \gamma$ . By assumption, we have  $s_\gamma s_\beta s_\alpha(\chi) \leq 0$ .

So  $(a_\beta + a_\gamma - a_\alpha)\alpha + (a_\beta - a_\alpha)\gamma + (a_\gamma - a_\alpha)\beta \leq 0$ .

$$\text{Hence, we have } a_\beta + a_\gamma \leq a_\alpha \longrightarrow (7)$$

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\beta} \rangle \geq 0$  and  $\langle \chi, \check{\gamma} \rangle \geq 0$ .

$$\text{So we have, } a_\alpha \leq 2a_\beta \text{ and } a_\alpha \leq 2a_\gamma \longrightarrow (8).$$

Using (7) and (8),  $2a_\alpha \geq 2(a_\beta + a_\gamma) \geq 2a_\alpha$ . This is possible only if  $2a_\beta = a_\alpha = 2a_\gamma$ .

Then,  $\chi$  must be of the form  $a(\beta + 2\alpha + \gamma)$ , for some  $a \in \mathbb{Z}_{\geq 0}$ . □

Now for given an irreducible root system  $R$ , we describe all the Coxeter elements  $w \in W$  for which there is a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ . For the Dynkin diagrams and labelling of simple roots, we refer to page 58 of [2].

**Theorem 4.2.** (A) Type  $A_n$ : (1)  $A_3$ : For any Coxeter element  $w$ ,  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight.

(2)  $A_n, n \geq 4$ : If  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight and  $w$  is a Coxeter element, then  $w$  must be either  $s_n s_{n-1} \dots s_1$  or  $s_i \dots s_1 s_{i+1} \dots s_n$  for some  $1 \leq i \leq n-1$ .

(B) Type  $B_n$ : (1)  $B_2$ : For any Coxeter element  $w$ ,  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight.

(2)  $B_n, n \geq 3$ : If  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight and  $w$  is a Coxeter element, then  $w = s_n s_{n-1} \dots s_1$ .

(C) Type  $C_n$ : If  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight and  $w$  is a Coxeter element, then  $w = s_n s_{n-1} \dots s_1$ .

(D) Type  $D_n$ : (1)  $D_4$ :  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight and  $w$  is a Coxeter element if and only if  $l(ws_2) = l(w) + 1$ .

(2)  $D_n, n \geq 5$ : If  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for some non-zero dominant weight and  $w$  is a Coxeter element, then  $w = s_n s_{n-1} \dots s_1$ .

(E)  $E_6, E_7, E_8$ : There is no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$ .

(F)  $F_4$ : There is no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$ .

(G)  $G_2$ : There is no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$ .

*Proof.* By lemma (2.1),  $X(w)_T^{ss}(\mathcal{L}_\chi \neq \emptyset$  for a non-zero dominant weight  $\chi$  if and only if  $w\chi \leq 0$ . So, using this lemma we investigate all the cases.

### **Proof of (A):**

(1) The Coxeter elements of  $A_3$  are precisely  $s_1 s_2 s_3, s_1 s_3 s_2, s_2 s_1 s_3, s_3 s_2 s_1$ . For  $w = s_1 s_3 s_2$ , take  $\chi = \alpha_1 + 2\alpha_2 + \alpha_3$ . Otherwise take  $\chi = \alpha_1 + \alpha_2 + \alpha_3$ . Then  $w\chi \leq 0$ .

(2) Let  $n \geq 4$ , and let  $w\chi \leq 0$  for some dominant weight  $\chi$ . By lemma (4.1), if  $l(ws_i) = l(w) - 1$ , then  $i = 1$  or  $i = n$ .

If  $l(ws_n) \neq l(w) - 1$ , then using the fact that  $s_i$  commute with  $s_j$  for  $j \neq i-1, i+1$ , it is easy to see that  $w = s_n s_{n-1} \dots s_2 s_1$ .

If  $l(ws_n) = l(w) - 1$ , then, let  $i$  be the least integer in  $\{1, 2, \dots, n-1\}$  such that  $w = \phi s_{i+1} \dots s_n$ , for some  $\phi \in W$  with  $l(w) = l(\phi) + (n-i)$ . Then, we have to show that  $\phi = s_i s_{i-1} \dots s_1$ .

If  $\phi = \phi_1 s_j$  for some  $j \in \{2, 3, \dots, i-1\}$ , then  $w$  is of the form

$$\begin{aligned} w &= \phi_1 s_j (s_{i+1} \dots s_{n-1} s_n) \\ &= \phi_1 (s_{i+1} \dots s_{n-1} s_n s_j). \end{aligned}$$

This contradicts lemma (4.1). So  $j \in \{1, i\}$ . Again  $j = i$  is not possible unless  $i = 1$  by the minimality of  $i$ .

Thus, we have  $\phi = s_i \dots s_1$ .

**Proof of (B):**

(1) For  $w = s_1 s_2$ , take  $\chi = \alpha_1 + 2\alpha_2$ .

For  $w = s_2 s_1$ , take  $\chi = \alpha_1 + \alpha_2$ .

(2) For  $w = s_n s_{n-1} \dots s_1$ , take  $\chi = \alpha_1 + \alpha_2 + \dots \alpha_n$ . Then  $w\chi = -\alpha_n \leq 0$ .

Conversely, let  $w$  be a Coxeter element and let  $\chi$  be a non-zero dominant weight such that  $w\chi \leq 0$ . By lemma (4.1), if  $l(ws_i) = l(w) - 1$  then either  $i = 1$  or  $i = n$ .

If  $l(ws_n) \neq l(w) - 1$ , then using the fact that  $s_i$  commute with  $s_j$  for  $j \neq i - 1, i + 1$ , it is easy to see that  $w = s_n s_{n-1} \dots s_2 s_1$ .

We now claim that  $l(ws_n) = l(w) + 1$ . If not, then, the coefficient of  $\alpha_n$  in  $w\chi =$  coefficient of  $\alpha_n$  in  $s_n \chi$ .

Now, the coefficient of  $\alpha_n$  in  $s_n \chi$  is  $2a_{n-1} - a_n$ . Since  $w\chi \leq 0$ , we have  $2a_{n-1} - a_n \leq 0$ .

$$\Rightarrow 2a_{n-1} \leq a_n. \quad \longrightarrow (1)$$

Since  $\chi$  is dominant, we have  $\langle \chi, \alpha_{n-1}^\vee \rangle \geq 0$ . Thus, we get

$$-a_{n-2} + 2a_{n-1} - a_n \geq 0.$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - a_n \leq 0, \text{ by (1).}$$

So  $a_{n-2} = 0$ , a contradiction to the assumption that  $n \geq 3$  and  $\chi$  is a non-zero dominant weight. Thus  $l(ws_n) = l(w) + 1$ .

So the only possibility for  $w$  is  $s_n s_{n-1} \dots s_1$ .

**Proof of (C):**

For  $w = s_n s_{n-1} \dots s_1$ , take  $\chi = 2(\sum_{i \neq n} \alpha_i) + \alpha_n$ . Then,  $\chi$  is dominant and  $w\chi = -\alpha_n$ .

Conversely, let  $w$  be a Coxeter element and let  $\chi$  be a non-zero dominant weight such that  $w\chi \leq 0$ . By lemma (4.1), if  $l(ws_i) = l(w) - 1$  then  $i \in \{1, n\}$ .

If  $l(ws_n) \neq l(w) - 1$ , then using the fact  $s_i$  commute with  $s_j$  for  $j \neq i - 1, i + 1$ , it is easy to see that  $w = s_n s_{n-1} \dots s_2 s_1$ .

*Claim:*  $l(ws_n) = l(w) + 1$ .

If not, then, the coefficient of  $\alpha_n$  in  $w\chi =$  coefficient of  $\alpha_n$  in  $s_n \chi$ .

Now, the coefficient of  $\alpha_n$  in  $s_n \chi$  is  $a_{n-1} - a_n$ . Since  $w\chi \leq 0$ , we have  $a_{n-1} - a_n \leq 0$ .

$$\text{Hence, we have } a_{n-1} \leq a_n. \quad \longrightarrow (2)$$

Since  $\chi$  is dominant, we have  $\langle \chi, \alpha_{n-1}^\vee \rangle \geq 0$ . Thus, we get

$$-a_{n-2} + 2a_{n-1} - 2a_n \geq 0.$$

$$\Rightarrow a_{n-2} \leq 2a_{n-1} - 2a_n \leq 0, \text{ by (2).}$$

So  $a_{n-2} = 0$ , a contradiction to the assumption that  $\chi$  is a non-zero dominant weight.

Thus  $l(ws_n) = l(w) + 1$ .

So the only possibility for  $w$  is  $s_n s_{n-1} \dots s_1$ .

**Proof of (D):**

(1) For  $w = s_4 s_3 s_2 s_1$ , take  $\chi = 2(\alpha_1 + \alpha_2) + \alpha_3 + \alpha_4$ , for  $w = s_4 s_1 s_2 s_3$ , take  $\chi = 2(\alpha_3 + \alpha_2) + \alpha_1 + \alpha_4$  and for  $w = s_3 s_1 s_2 s_4$ , take  $\chi = 2(\alpha_4 + \alpha_2) + \alpha_1 + \alpha_3$ .

The converse follows from lemma (4.1).

(2) For  $w = s_n s_{n-1} \dots s_1$ , take  $\chi = 2(\sum_{i=1}^{n-2} \alpha_i) + \alpha_{n-1} + \alpha_n$ . Then  $w\chi \leq 0$ .

Conversely, let  $w$  be a Coxeter element and let  $\chi$  be a non-zero dominant weight such that  $w\chi \leq 0$ . By lemma (4.1), if  $l(ws_i) = l(w) - 1$  then  $i \in \{1, n-1, n\}$ .

Now, if  $l(ws_1) = l(w) - 1$ , then, it is easy to see that  $w = s_n s_{n-1} \dots s_2 s_1$ .

So, it is sufficient to prove that  $l(ws_n) = l(w) + 1$  and  $l(ws_{n-1}) = l(w) + 1$ .

If  $l(ws_n) = l(w) - 1$ , then, the coefficient of  $\alpha_n$  in  $w\chi = \text{coefficient of } \alpha_n \text{ in } s_n \chi = a_{n-2} - a_n$ .

Since  $w\chi \leq 0$ , we have  $a_{n-2} - a_n \leq 0$ .  $\longrightarrow (4)$

Since  $\chi$  is dominant we have  $\langle \chi, \alpha_{n-2}^\vee \rangle \geq 0$ . Therefore, we have

$$2a_{n-2} \geq a_{n-1} + a_{n-3} + a_n. \longrightarrow (5)$$

Also, since  $\langle \chi, \alpha_{n-1}^\vee \rangle \geq 0$  and  $\langle \chi, \alpha_{n-3}^\vee \rangle \geq 0$ , we have

$$2a_{n-1} - a_{n-2} \geq 0 \longrightarrow (6)$$

$$\text{and } 2a_{n-3} - a_{n-4} - a_{n-2} \geq 0. \longrightarrow (7)$$

From (5), we get

$$\begin{aligned} 4a_{n-2} &\geq 2a_{n-1} + 2a_{n-3} + 2a_n \\ &\geq a_{n-2} + (a_{n-4} + a_{n-2}) + 2a_n, \text{ from (6) and (7)} \\ &\geq 2a_{n-2} + 2a_{n-2} + a_{n-4}, \text{ by (4)} \\ &= 4a_{n-2} + a_{n-4}. \end{aligned}$$

So  $a_{n-4} = 0$ , a contradiction to the assumption that  $\chi$  is a non-zero dominant weight.

So  $l(ws_n) = l(w) + 1$ .

Using a similar argument, we can show that  $l(ws_{n-1}) = l(w) + 1$ .

**Proof of (E):**

**Type E<sub>8</sub> :**

Let  $w$  be a Coxeter element and let  $\chi$  be a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ . Further, if  $l(ws_i) = l(w) - 1$ , then by lemma (4.1),  $i \in \{1, 2, 8\}$ .

*Case 1 :  $i = 8$*

Co-efficient of  $\alpha_8$  in  $w\chi$  = Co-efficient of  $\alpha_8$  in  $s_8(\chi) = a_7 - a_8 \leq 0$ .

Since  $\chi$  is dominant,  $\langle \chi, \check{\alpha}_i \rangle \geq 0 \forall i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ .

$$\langle \chi, \check{\alpha}_7 \rangle \geq 0 \Rightarrow 2a_7 \geq a_6 + a_8 \geq a_6 + a_7.$$

Hence, we have  $a_7 \geq a_6$ .

$$\begin{aligned} \langle \chi, \check{\alpha}_6 \rangle \geq 0 &\Rightarrow 2a_6 \geq a_5 + a_7 \geq a_5 + a_6 \\ &\Rightarrow a_6 \geq a_5. \end{aligned}$$

$$\begin{aligned} \langle \chi, \check{\alpha}_5 \rangle \geq 0 &\Rightarrow 2a_5 \geq a_4 + a_6 \geq a_4 + a_5. \\ &\Rightarrow a_5 \geq a_4 \end{aligned}$$

$$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq a_1 + a_4.$$

$$\langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 \geq a_4.$$

$$\begin{aligned} \text{Now, } \langle \chi, \check{\alpha}_4 \rangle \geq 0 &\Rightarrow 2a_4 \geq a_2 + a_3 + a_5 \\ &\Rightarrow 4a_4 \geq 2a_2 + 2a_3 + 2a_5. \\ &\geq a_4 + a_1 + a_4 + 2a_4, \text{ since } a_5 \geq a_4. \end{aligned}$$

So,  $a_1 = 0$ . Thus in this case, there is no Coxeter element  $w$  for which there is a non-zero dominant weight such that  $w\chi \leq 0$ .

*Case 2 :  $i = 1$*

Co-efficient of  $\alpha_1$  in  $w\chi$  = Co-efficient of  $\alpha_1$  in  $s_1\chi = a_3 - a_1 \leq 0$ .

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\alpha}_3 \rangle \geq 0$ . Therefore,  $2a_3 \geq a_1 + a_4 \geq a_3 + a_4$

Hence, we have  $a_3 \geq a_4$ .

Since,  $\langle \chi, \check{\alpha}_4 \rangle \geq 0$ , we have  $2a_4 \geq a_3 + a_2 + a_5$ .

Since,  $\langle \chi, \check{\alpha}_2 \rangle \geq 0$  and  $\langle \chi, \check{\alpha}_5 \rangle \geq 0$  we have  $2a_2 \geq a_4$  and  $2a_5 \geq a_4 + a_6$ .

Then,  $4a_4 \geq 2a_3 + 2a_2 + 2a_5 \geq 2a_4 + a_4 + a_4 + a_6$ , from the above inequalities.

So,  $a_6 = 0$ . Hence we have  $\chi = 0$ . Thus, in this case also, there no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

*Case 3 :  $i = 2$*

Co-efficient of  $\alpha_2$  in  $w\chi$  = Co-efficient of  $\alpha_2$  in  $s_2\chi = a_4 - a_2 \leq 0$ .

Since  $\chi$  is dominant,  $\langle \chi, \check{\alpha}_i \rangle \geq 0 \forall i \in \{1, 2, 3, 4, 5, 6\}$ .

$$\langle \chi, \check{\alpha}_5 \rangle \geq 0 \Rightarrow 2a_5 \geq a_4 + a_6.$$

$$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq a_1 + a_4.$$

$$\langle \chi, \check{\alpha}_4 \rangle \geq 0 \Rightarrow 2a_4 \geq a_3 + a_2 + a_5.$$

Hence, we have  $4a_4 \geq 2a_3 + 2a_2 + 2a_5$ .



$$\geq (a_1 + a_4) + 2a_4 + (a_4 + a_6) = a_1 + a_6 + 4a_4.$$

$\Rightarrow a_1 + a_6 = 0$ . So,  $a_1 = a_6 = 0$ .

Hence, we have  $\chi = 0$ . Thus, in this case also, there no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

**Type  $E_6, E_7$  :**

Proof is similar to the case of  $E_8$ .

**Proof of  $F$ :**

Let  $w$  be a Coxeter element. Let  $\chi$  be a non-zero dominant weight such that  $w\chi \leq 0$ . If  $l(ws_i) = l(w) - 1$ , then  $i \in \{1, 4\}$ , by lemma (4.1).

*Case 1 :  $i = 1$*

Co-efficient of  $\alpha_1$  in  $w\chi =$  Co-efficient of  $\alpha_1$  in  $s_1\chi = a_2 - a_1 \leq 0$ .

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\alpha}_3 \rangle \geq 0$  and  $\langle \chi, \check{\alpha}_2 \rangle \geq 0$ .

$\langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_2 + a_3$ , since  $a_2 \leq a_1$ .

Hence, we have  $a_2 \geq a_3$ .

$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_3 + a_4$ .

So, we have  $a_4 = 0$ . Hence,  $\chi = 0$ . Thus, in this case there no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

*Case 2 :  $i = 4$*

Co-efficient of  $\alpha_4$  in  $w\chi =$  Co-efficient of  $\alpha_4$  in  $s_4\chi = a_3 - a_4 \leq 0$ .

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\alpha}_3 \rangle \geq 0$  and  $\langle \chi, \check{\alpha}_2 \rangle \geq 0$ .

$\langle \chi, \check{\alpha}_3 \rangle \geq 0 \Rightarrow 2a_3 \geq 2a_2 + a_4 \geq 2a_2 + a_3$ , since  $a_3 \leq a_4$ .

Hence, we have  $a_3 \geq 2a_2$ .

$\langle \chi, \check{\alpha}_2 \rangle \geq 0 \Rightarrow 2a_2 \geq a_1 + a_3 \geq a_1 + 2a_2$ .

So, we have  $a_1 = 0$ . Hence,  $\chi = 0$ . Thus, in this case also, there no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

**Proof of  $G$ :**

Let  $w$  be a Coxeter element and  $\chi = a_1\alpha_1 + a_2\alpha_2$ , be a dominant weight such that  $w\chi \leq 0$

*Case 1 :  $l(ws_1) = l(w) - 1$ .*

Co-efficient of  $\alpha_1$  in  $w\chi =$  Co-efficient of  $\alpha_1$  in  $s_1\chi = a_2 - a_1 \leq 0$ .

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\alpha}_2 \rangle \geq 0$ .

$$\Rightarrow 2a_2 \geq 3a_1 \geq 3a_2.$$

So, we have  $a_2 = 0$ . Hence,  $\chi = 0$ . Thus, in this case, there no Coxeter element  $w$  for

which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

*Case 2 :  $l(ws_2) = l(w) - 1$ .*

Co-efficient of  $\alpha_2$  in  $w\chi$  = Co-efficient of  $\alpha_2$  in  $s_2\chi = 3a_1 - a_2 \leq 0$ .

Since  $\chi$  is dominant, we have  $\langle \chi, \check{\alpha}_1 \rangle \geq 0$ .

$$\Rightarrow 2a_1 \geq a_2 \geq 3a_1.$$

So, we have  $a_1 = 0$ . Hence,  $\chi = 0$ . Thus, in this case also, there no Coxeter element  $w$  for which there exist a non-zero dominant weight  $\chi$  such that  $w\chi \leq 0$ .

□

## References

- [1] R.W. Carter, Finite Groups of Lie type, John Wiley, New York, 1993.
- [2] J.E. Humphreys, Introduction to Lie algebras and representation theory, Springer, Berlin Heidelberg, 1972.
- [3] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Univ. Press, Cambridge, 1990.
- [4] S. S. Kannan, Torus quotients of homogeneous spaces, Proc. Indian Acad. Sci.(Math. Sci), 108(1998), no 1, 1-12.
- [5] S. S. Kannan, Torus quotients of homogeneous spaces-II, Proc. Indian Acad. Sci.(Math. Sci), 109(1999), no 1, 23-39.
- [6] S.S. Kannan, Cohomology of line bundles on Schubert varieties in the Kac-Moody setting, J. Algebra, 310(2007) 88–107.
- [7] S. S. Kannan, Pranab Sardar, Torus quotients of homogeneous spaces of the general linear group and the standard representation of certain symmetric groups, to appear in Proc. Indian Acad. Sci.
- [8] D. Mumford, J. Fogarty and F. Kirwan, Geometric Invariant theory, (Third Edition), Springer-Verlag, Berlin Heidelberg, New York, 1994.
- [9] P.E. Newstead, Introduction to Moduli Problems and Orbit Spaces, TIFR Lecture Notes, 1978.
- [10] C.S. Seshadri, Quotient spaces modulo reductive algebraic groups, Ann. Math. 95(1972) 511-556.
- [11] C.S. Seshadri, Introduction to Standard Monomial theory, Lecture notes No.4, Brandeis University, Waltham, MA, 1985.